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FORMATION OF FOCUSSED SPACE-CHARGE-LIMITED ELECTRONS AND ION BEAMS

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DEFINITION OF CERTAIN SYMBOLS

v (Introduction and Section 1) = velocity in cm/sec.

v (Sections 4, 5, and 6 and Tables) = Contraction parameter, dimensionless ratio.

V (Introduction and Sections 1 and 2) = Potential (kilovolts).

V (Sections 3, 4, 5, and 6) = Potential (arbitrary units).

u (Section 2) = cathode distance (units of radius).

u (Sections 4, 5, and 6, Tables, and Figures 6 to 10, inclusive) =abscissa scale parameter, dimensionless ratio.

U (Section 3) = Potential function of x (units arbitrary).

U (Section 4) = Potential function of u (units arbitrary).

G(u) (Section 2) = "Solving" function of differential equation.

G(u) (Sections 4 and 5 and Tables) = Potential function (units arbitrary).

k (Section 3) = Parameter (dimensionless).

K (Introduction and Sections 1 and 2) = Constant.

FORMATION OF FOCUSSED SPACE-CHARGE-LIMITED

ELECTRONS AND ION BEAMS

By O. Bunemann, E. U. Condon, and A. Latter

This report deals with the initial acceleration of intense beams of electrons or ions in such a way as to produce focussed beams. The theory is developed for the cylindrical case in which the emitter is long in one direction. The discussion will be written out explicitly for electrons, but is applicable with obvious changes to space-charge-limited positive ion beams. The object of the calculations is to learn how to design the emitting surface and the related electrodes to produce narrowly focussed cathode ray beams. The basic arrangement under consideration is sketched in Figure 1.

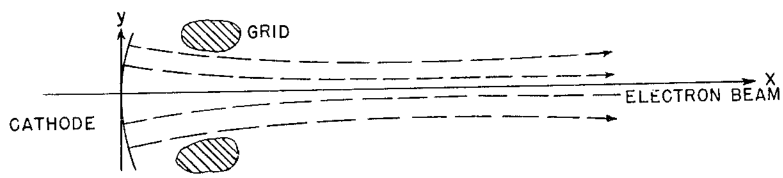

Figure 1.

Potential is reckoned as zero at the cathode and the initial velocity of emission of electrons from the cathode is neglected. At a place where the potential is V (kilovolts) the velocity of an electron is

$$v = \sqrt{\frac{2eV \cdot 100}{300m}} = 1.87 \times 10^9 \sqrt{V} \quad \text{cm/sec} \tag{1}$$

provided, of course, $V \ll 500$ so that relativistic variation of mass with velocity is negligible.

Let I (amp/cm^2) be the electron current density. Then since the charge on one electron is 1.60 x 10^{-19} coulombs, the density of electrons in a beam is given by

$$\eta = I/ev = 3.34 \times 10^9 \, I/\sqrt{V} \text{ electrons/cm}^3 \tag{2}$$

and the space charge density due to these electrons is then

$$\rho = \eta e = -1.60 \, I/\sqrt{V} \text{ esu/cm}^3 \tag{3}$$

Poisson's equation for the potential becomes, in these units,

$$\nabla^2 V = -1.2\pi\rho \tag{4}$$

so that if ρ is given by the preceding expression, the form assumed by Poisson's equation is

$$\nabla^2 V = K I/\sqrt{V} \tag{5}$$

in which K = 1.2π x 1.60 = 6.0. Going back through the steps we see that the fundamental formula for K is

$$K = 1.2\pi \times 3 \times 10^9 \sqrt{\frac{0.3m}{2\ e}} \tag{6}$$

in which m is the mass of the electron or ion in grams and e is its charge in electrostatic units.

1. PLANE PARALLEL BEAM

First suppose that the cathode is in the plane x = 0 and that all electrons move out normally to the cathode so that I is constant. Then the problem reduces to the familiar one-dimensional problem

$$\frac{d^2V}{dx^2} = K \frac{1}{\sqrt{V}} \tag{5a}$$

of which the solution for space charge limited currents (V = 0 and dV/dx = 0 at x = 0) is

$$V = (\frac{9}{4} K\ I)^{2/3}\ x^{4/3} \tag{7}$$

Since $(9/4\ K)^{2/3} = 5.7$ it follows that the potential must be 5.7 kv at a distance of one centimeter from a cathode that is emitting a space charge limited electron beam of 1 amp/cm^2.

In practice the cathode will be of finite extent. Suppose the current density is I for $y < 0$ but is zero for $y > 0$ so the plane y = 0 is the upper edge of the beam. In the region $y > 0$ above the beam the potential must satisfy Laplace's equation and on the boundary at y = 0 must assume the values given by equation 7. Evidently the cathode extends upward beyond the emitting area as a continuation of the x = 0 plane, then the electrons near the upper edge of the beam will be repelled by those deeper in the beam, causing the beam to spread upward into the y = 0 region. By attaching a wing to the cathode, nonemitting but at cathode potential, one can produce an electric field which counteracts this tendency and results in a parallel beam.

The solution of Laplace's equation which has the desired values along the boundary is given by

$$V = (\frac{9}{4} K\ I)^{2/3}\ r^{4/3} \cos \frac{4}{3}\theta \tag{8}$$

where (r, θ) are polar coordinates as indicated in Figure 2. Accordingly, the potential must vanish not only at x = 0 but at $4/3\ \theta = \pi/2$ that is

$$\theta = 3\pi/8 = 67.5° \tag{9}$$

which is the angle used in Figure 2 for the cathode wing.

This result for the proper wing to keep the beam from spreading is not new. It was first obtained by Peirce (Journal of Applied Physics). It is given here for convenient reference and by way of warming up to the rest of the calculations, which are believed to be new.

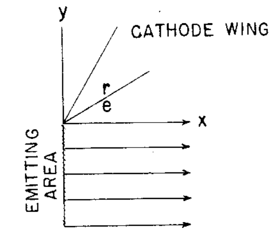

Figure 2.

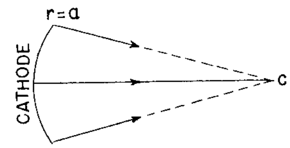

Figure 3.

2. BEAM CONVERGING TO THE FOCUS

Suppose now, we have a concave (cylindrically) cathode and wish to have an electron beam start radially inward toward a focus at the center of curvature of the cathode. Because of the convergence of the trajectories, the current density at the cathode, $r = a$, increases as one goes toward the focus. Right at the focus, if the same radial motion continued the charge density would be infinite, so the potential would be infinite. In practice, therefore, where only finite potentials are applicable, the beam will start to spread before it reaches the center (Figure 3).

Using cylindrical polar coordinates with the center at the center of curvature, let $r = a$ be the emitting surface. Let I be the current density at $r = a$ so that $I \cdot (a/r)$ is the current density at r, so long as the electrons have continued to move radially inward. In polar coordinates the equation for the potential becomes

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) = K\frac{Ia}{\sqrt{V}}\frac{1}{r} \tag{10}$$

We have to find a particular solution of this for which $V = 0$ and $dV/dr = 0$ at $r = a$. Let $\rho = r/a$ then the equation 10 becomes

$$\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right) = \frac{KIa^2}{\sqrt{V}} \tag{11}$$

The distance from the cathode measured in units of the radius is

$$u = (1-\rho) \tag{12}$$

in terms of which

$$\frac{d}{du}\left[(1-u)\frac{dV}{du}\right] = \frac{KIa^2}{V} \tag{13}$$

It is natural to look for a solution in the form of equation 7 multiplied by a factor which corrects for the convergence of the beam,

$$V(u) = \left(\frac{9}{4}KI^2\right)^{2/3} u^{4/3}\ F(u) \tag{14}$$

This leads to the following differential equation for F (u),

$$u^2(1-u)F'' + u\left(\frac{8}{3}-\frac{11}{3}u\right)F' + \left(\frac{4}{9}-\frac{10}{9}u\right)F = \frac{4}{9}\frac{1}{\sqrt{F}} \tag{15}$$

The substitution $F = 1/G^2$ leads to a differential equation for G(u) of which the particular solution

$$G(u) = 1 - \frac{4}{15}u - \frac{7}{90}u^2 - \frac{23}{675}u^3 - \frac{22}{945}u^4 \qquad (16)$$

is the one desired.

This leads to the following brief table of values for F(u)

u	F(u)
0	1.00
0.2	1.18
0.4	1.43
0.6	1.82

From this we see that, for example, at a distance 0.4 of the cathode radius in from the cathode, the potential must be 1.43 times what is required in the plane parallel case to compensate for the higher density of space charge resulting from the convergence of the beam.

In the same way, one could calculate the amount by which the potential needed is less, in case the beam spreads radially outward from a convex emitting surface. No explicit calculations for this case have been made for this report.

Another consequence of the convergence of the beam is that the cathode wing at the edge of the beam must have a different shape in order to provide the proper field at the edge to keep the electrons moving radially inward. As before, the wing must start off making an angle of 67.5° with the edge of the beam, but in this case, it must curve inward toward the beam instead of remaining plane. A rough calculation of this was made and is shown in Figure 4. The method is as follows: From equation 16 we can get a power series for F(u)

$$F(u) = 1 + \frac{8}{15}u + \ldots$$

Hence, the values assumed by the potential along the edge of the beam are given by

$$V = \left(\frac{9}{4}KIa^2\right)^{2/3} u^{\frac{4}{3}}\left[1 + \frac{8}{15}u + \ldots\right] \qquad (17)$$

Taking polar coordinates at the edge of the beam (with r in the same unit as u) as in Figure 2, the appropriate solution for V in the region outside the beam is

$$V = \left(\frac{9}{4}KIa^2\right)^{2/3}\left[r^{4/3}\cos\frac{4}{3}\theta + \frac{8}{15}r^{7/3}\cos\frac{7}{3}\theta + \ldots\right] \qquad (18)$$

The equation for the cathode wing is obtained by equating the potential to zero. The rough curve of Figure 4 was obtained from equation 18 using only the two terms explicitly written. If more accuracy is needed, it could be obtained by carrying equation 17 and hence equation 18 to more terms.

3. BEAMS OF SMALL ANGULAR SPREAD

We now develop a method for calculating the proper electrode shapes to be used in obtaining space charge limited electron beams of any type so long as all the trajectories make small angles with a central axis. Suppose we have a situation as in Figure 5 where all the trajectories start from

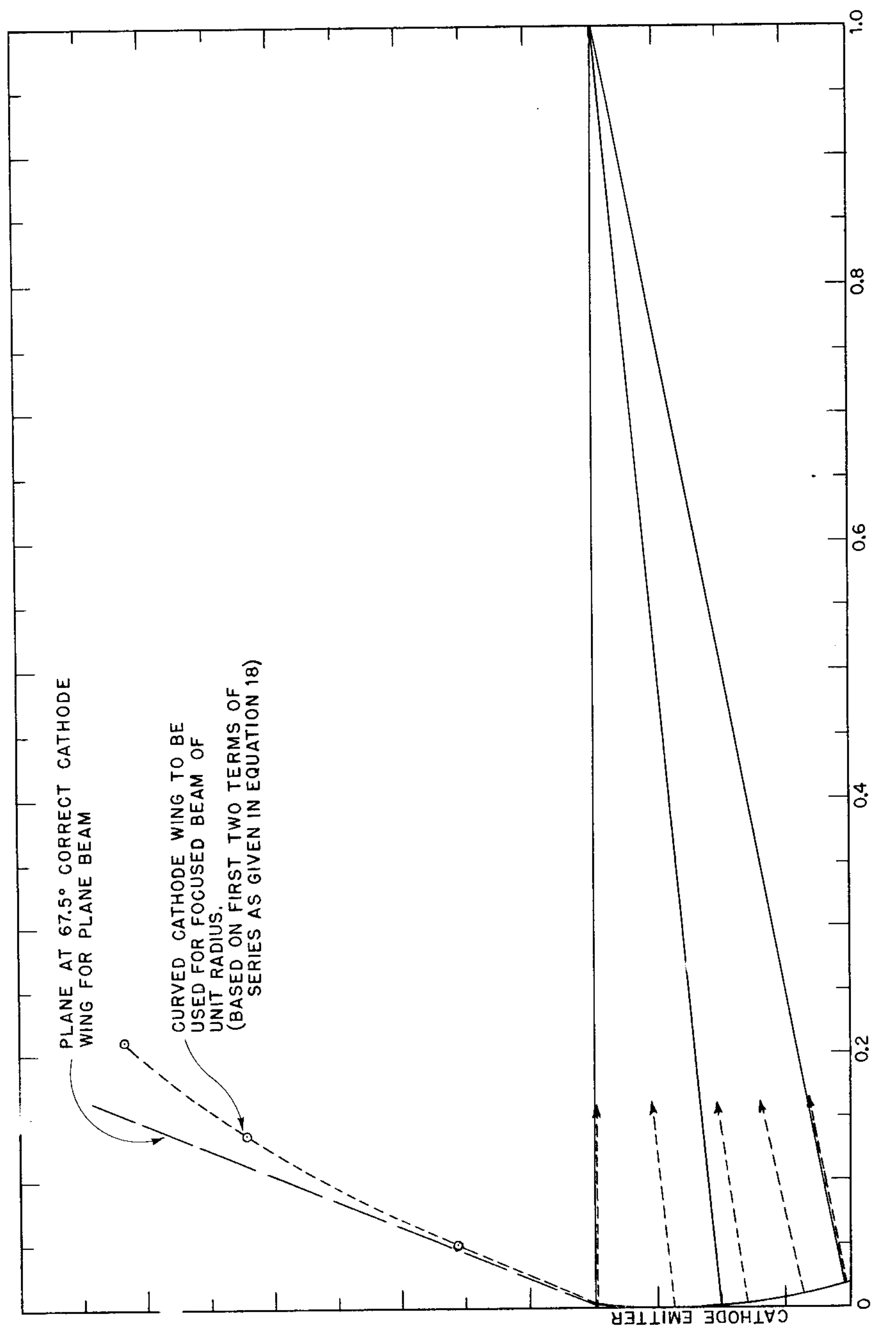
PLANE AT 67.5° CORRECT CATHODE WING FOR PLANE BEAM
CURVED CATHODE WING TO BE USED FOR FOCUSED BEAM OF UNIT RADIUS. (BASED ON FIRST TWO TERMS OF SERIES AS GIVEN IN EQUATION 18)
CATHODE EMITTER
0
0.2
0.4
0.6
0.8
1.0

Figure 4.

an emitting strip at $x = 0$ and move in such a way that $(dy/dx)^2 \ll 1$ at all points of every trajectory. The problem to be solved is to develop a method for calculating electrode shapes which will result in a space charge limited beam for which the trajectories are given.

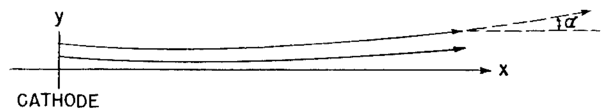

Figure 5.

Let $Y(x)$ be the equation of the trajectory at the upper edge of the beam. It will be supposed that $y = 0$ is also a trajectory and that the lower edge of the beam is given by the symmetrical trajectory, $-Y(x)$. It will also be supposed that all the intermediate trajectories are given by $y = k\,Y(x)$ where k is a parameter which varies in the range $-1 \leq k \leq +1$. Moreover, it will be supposed that the current is equally distributed over the cross section of the beam so that the current between the k trajectory and the k + dk trajectory is $\frac{1}{2}J dk$ where J is the total current in the beam, per unit length in the z-direction.

Letting $V(x,y)$ be the actual potential in the beam as it exists with due regard to the effects of space charge, the equations of motion for an electron of charge $-e$ and mass m become

$$\frac{d^2x}{dt^2} = \frac{e}{m}\frac{\delta V}{\delta x}$$
$$\frac{d^2y}{dt^2} = \frac{e}{m}\frac{\delta V}{\delta y} \qquad (19)$$

Let the potential along the axis be denoted by $U(x)$ so

$$V(x,0) = U(x)$$

then, because of symmetry around $y = 0$ we have $\frac{\delta V}{\delta y} = 0$ at $y = 0$. Because of the narrowness of spread of the beam, we may approximate to the potential as follows:

$$V(x,y) = U(x) + \tfrac{1}{2}\, y^2 W(x) \qquad (20)$$

The motion in x of the axial ray will be thus given by

$$\frac{d^2x}{dt^2} = \frac{e}{m}\frac{\delta U(x)}{\delta x}$$

hence

$$\left(\frac{dx}{dt}\right)^2 = \frac{2e}{m}\,U(x) \qquad (21)$$

It will be assumed that this equation is also applicable to the x motion of the other rays with sufficient accuracy. The motion in y of the rays is given by

$$\frac{d^2y}{dt^2} = \frac{e}{m}\,y \cdot W(x) \qquad (22)$$

Since this equation is linear in y, this justifies to this approximation the assumption that if Y(x) is a trajectory, kY(x) is also a trajectory, provided the initial conditions are suitably arranged.

We may use equation 21 to eliminate t from equation 22 and obtain a differential equation for trajectory Y(x);

$$\sqrt{\frac{2e}{m}U}\,\frac{d}{dx}\left(\sqrt{\frac{2e}{m}U}\,\frac{dy}{dx}\right) = \frac{e}{m}\,y \cdot W(x)$$

or

$$\sqrt{U}\,\frac{d}{dx}\left(\sqrt{U}\,\frac{dy}{dx}\right) = \tfrac{1}{2}\,y \cdot W(x) \tag{23}$$

Next we must discuss the function W(x) which was introduced in equation 20. The current density at x is J/2Y(x) and the charge density is therefore

$$\rho = -\frac{J}{2Y(x)\sqrt{\frac{2e}{m}U}} \tag{24}$$

where it is supposed that J is in esu of current per unit length. The Poisson equation for the potential is thus

$$\frac{\delta^2 V}{\delta x^2} + \frac{\delta^2 V}{\delta y^2} = \frac{2\pi J}{Y(x)\sqrt{\frac{2e}{m}U}}$$

Using equation 20 to evaluate the derivatives along y = 0 we have

$$\frac{\delta^2 V}{\delta x^2} = U''(x) \text{ and } \frac{\delta^2 V}{\delta y^2} = W(x)$$

and therefore the Poisson equation is

$$U''(x) + W(x) = \frac{2\pi J}{Y(x)\sqrt{\frac{2e}{m}U}} \tag{25}$$

which permits us to eliminate W(x) from equation 23 and so to write the y equation of motion entirely in terms of U and its derivatives:

$$\sqrt{U}\frac{d}{dx}\left(\sqrt{U}\,\frac{dy}{dx}\right) + \tfrac{1}{2}\,U''(x)y = \frac{\pi J}{\sqrt{\frac{2e}{m}U}}\,\frac{y}{Y(x)} \tag{26}$$

Now we observe that y = Y(x) is, by hypothesis, a solution of equation 26 regarded as a differential equation for Y(x). Hence if we substitute an assumed analytical form y = Y(x) for the desired limiting ray of the beam, we get a differential equation which has to be satisfied by U(x), the potential along the axis of the beam. It is

$$Y(x)\,U''(x) + Y'(x)U'(x) + 2Y''(x)\,U(x) = \frac{2\pi J}{\sqrt{\frac{2e}{m}U}} \tag{27}$$

In the special case of the plane parallel beam, we have Y(x) = 1 and equation 27 reduces to equation 5a which has already been discussed in Section 1, the current per unit area being $\frac{1}{2}J = I$. Similarly in the case of the convergent beam discussed in Section 2, we have $Y(x) = 1 - \frac{x}{a}$ and equation 27 reduces exactly to equation 13 as already discussed. Sometimes it will be convenient to work with equation 27 in the form

$$\frac{d}{dx}\left\{Y(x)U'(x)\right\} + 2Y''(x)U(x) = \frac{KI}{\sqrt{U}} \text{ where } K = 4\pi\sqrt{m/2e} \tag{27a}$$

We have to solve this equation for U(x) under the conditions $U(x) = 0$ and $U'(x) = 0$ at $x = 0$ in order to find the required potential variation along the axis of a beam whose total current per unit length is I and whose edge is at $y = \pm Y(x)$.

4. BEAM WITH HYPERBOLIC TRAJECTORIES

A simple analytic form for the trajectories which resembles those sketched qualitatively in Figure 5 is represented by the family of hyperbolas

$$Y(x) = y_m + (y_0 - y_m) \cdot \frac{\left(\frac{x - x_m}{x_m}\right)^2}{1 + \frac{y_0 - y_m}{x_m} \frac{x}{x_m} \cot \alpha} \tag{28}$$

in which the symbols have the following meanings:

y_0 is the value of y at $x = 0$,

x_m, y_m are the coordinates of the place of narrowest section of the beam, and α is the angle which the extreme ray makes with the beam axis at large values of x (semi-angle of flare of the beam).

The initial width of the beam being $2y_0$, it follows that the initial current density (per unit area) in the beam cathode is

$$I_0 = J/2y_0 \quad \text{and } I_m = J/2y_m$$

is the maximum current density at the place of minimum section. The formula, equation 8 includes a parabolic trajectory as a limiting case if we put $\alpha = \pi/2$ so $\cot \alpha = 0$. It will be convenient to introduce certain reduced variables in equation 28 in place of the parameters occurring there. We write

$$u = \frac{x - x_m}{x_m} \tag{29}$$

so the beam starts at $u = -1$, and $u = 0$ corresponds to the place of minimum section. Also

$$v = \frac{y}{(y_0 - y_m)} \quad v_m = \frac{y_m}{(y_0 - y_m)} \tag{30}$$

so the unit in which transverse motion is measured is in terms of the constriction, $(y_0 - y_m)$. It follows that the constriction ratio y_m/y_0 is given by

$$y_m/y_0 = \frac{v_m}{1 + v_m}$$

hence a beam which does not constrict corresponds to $v_m \rightarrow \infty$ and, for example, $v_m = 1$ corresponds to a beam whose minimum section is one-half the section at the cathode. In terms of these units, equation 28 becomes

$$v = v_m + \frac{u^2}{1 + E(u+1)} \tag{31}$$

in which

$$E = \frac{y_0 - y_m}{x_m} \cot \alpha \tag{32}$$

See Figure 10 for several examples of this family of curves. A trajectory which has a slope, $-(E+2)\cdot(\frac{y_0-y_m}{x_m})$, at the cathode, has a slope, $\frac{1}{E}(\frac{y_0-y_m}{x_m})$, at large distances from the cathode, In other words, the ratio of slope at large distances to initial slope of any trajectory is: $-\frac{1}{E(E+2)}$, which shows how the parameter E is independent of the actual value of y_0.

Summarizing, we see that a beam is characterized by four parameters. These are v_m and E which are dimensionless and fix the contraction and flare, respectively, x_m which fixes the abscissa scale, and y_0 which fixes the actual width of the cathode. It is also convenient to write

$$U(u) = [KI_0 v_m]^{2/3} G(u)$$

where

$$G(u) = (1+v_m)^{2/3} H(u) \tag{33}$$

in which case equation 27 transforms into a simple canonical form for H(u)

$$\sqrt{H}\left[V(u)\frac{d^2H}{du^2} + \frac{dv}{du}\frac{dH}{du} + 2\frac{d^2v}{du^2}\right] = 1 \tag{34}$$

We need to have a particular solution of this equation such that $H(-1) = 0$ and $H'(-1) = 0$ corresponding to space charge limitation at the cathode. The functions v(u) and $\frac{dv}{du}$ and $\frac{d^2v}{du^2}$ are to be expressed by use of equation 31. Since equation 31 contains two parameters, v_m and E, there will be a two-parameter family of solutions of equation 34 that is required. Quite a number of these have been computed by numerical intergration of equation 34, and the results are presented in Tables 1, 2, 3, and 4.

It is important to recapitulate: The potential on the axis of a space charge limited electron beam whose current density is I_0 at the cathode and whose shape is given by equations 28 or 31 is given by equation 33 in terms of the special functions G(u), depending on the parameters v_m and E, whose values are given in Tables 1 to 4.

PHYSICAL DISCUSSION OF RESULTS FOR HYPERBOLIC TRAJECTORIES

The results obtained in the preceding section will now be discussed from the point of view of the main physical relations involved. It will be found that the family of hyperbolic trajectories, for different values of v_m and E covers a wide range of interesting cases.

Let us first consider the case $E \neq 0$ but $v_m \to \infty$. From equation 30 we see that this implies $y_m = y_0$ and so corresponds to a beam which does not contract at all before it starts to spread.

The limiting angle of semi-flare of the beam, α, is given by

$$\tan\alpha = \frac{y_0-y_m}{x_m}\cdot\frac{1}{E} = \frac{(\tan\alpha_0)}{E}$$

where

$$\tan\alpha_0 = \left(\frac{y_0-y_m}{x_m}\right) \tag{35}$$

Hence $v_m = \infty$ corresponds to a plane parallel beam for any value of E. This requires that G(u) in the limit $v_m \to \infty$ go over to the limit

$$G(u) \to \left(\frac{9}{4}\right)^{2/3}(u+1)^{4/3} \tag{36}$$

Table 1. G(u) for E = 2.

(u + 1)	$v_m = 2$	$v_m = 1$	$v_m = 1/2$	$v_m = 0$
0	.000	.000	.000	
0.1	.084	.086	.089	
0.2	.219	.227	.238	
0.3	.379	.397	.419	
0.4	.556	.582	.612	
0.5	.743	.772	.804	
0.6	.935	.963	.983	
0.7	1.133	1.151	1.145	
0.8	1.332	1.333	1.285	
0.9	1.532	1.506	1.404	
1.0	1.733	1.675	1.505	
1.1	1.934	1.837	1.590	
1.2	2.134	1.990	1.662	
1.3	2.333	2.140	1.725	
1.4	2.532	2.281	1.784	
1.5	2.731	2.420	1.840	
1.6	2.928	2.555	1.895	
1.7	3.124	2.690	1.950	
1.8	3.319	2.821	2.006	
1.9	3.515	2.949	2.066	
2.0	3.711	3.077	2.127	
2.1	3.905	3.201	2.193	
2.2	4.096	3.325	2.260	
2.3	4.291	3.449	2.333	
2.4	4.483	3.572	2.406	
2.5	4.675	3.694	2.484	
2.6	4.867	3.818	2.565	
2.7	5.056	3.940	2.646	
2.8	5.247	4.064	2.602	
2.9	5.438	4.187	2.691	
3.0	5.630	4.312	2.783	
3.1				
3.2				
3.3				
3.4				
3.5				
3.6				
3.7				

Table 2. G(u) for E = 1.

(u + 1)	$v_m = 2$	$v_m = 1$	$v_m = 1/2$	$v_m = 0$
0.0	.000	.000	.000	
0.1	.083	.085	.087	
0.2	.216	.226	.236	
0.3	.377	.400	.422	
0.4	.559	.593	.630	
0.5	.751	.797	.844	
0.6	.950	1.000	1.048	
0.7	1.152	1.198	1.226	
0.8	1.353	1.383	1.369	
0.9	1.551	1.552	1.470	
1.0	1.744	1.702	1.529	
1.1	1.931	1.834	1.552	
1.2	2.110	1.947	1.546	
1.3	2.284	2.046	1.517	
1.4	2.449	2.128	1.476	
1.5	2.609	2.196	1.431	
1.6	2.763	2.258	1.385	
1.7	2.909	2.309	1.336	
1.8	3.049	2.353	1.301	
1.9	3.184	2.396	1.271	
2.0	3.316	2.430	1.248	
2.1	3.441	2.465	1.221	
2.2	3.564	2.497	1.215	
2.3	3.681	2.528	1.230	
2.4	3.797	2.560	1.240	
2.5	3.908	2.589	1.256	
2.6	4.018	2.622	1.279	
2.7	4.125	2.652	1.307	
2.8	4.231	2.685	1.342	
2.9	4.334	2.720	1.379	
3.0	4.437	2.756	1.423	
3.1				
3.2				
3.3				
3.4				
3.5				
3.6				
3.7				

to be in agreement with the elementary solution for this case given in Section 1. This limiting form also occurs for $E \to \infty$ with v_m finite.

Table 5 gives the values of this limit for convenient comparison with the other cases.

The qualitative features of the potential variation along the axis of the beam are shown in Figures 6, 7, 8, 9. For example, Figures 6 and 7 show the case in which the minimum section of the beam is

Table 3. G(u) for E = 1/2.

(u + 1)	$v_m = 2$	$v_m = 1$	$v_m = 1/2$	$v_m = 0$
0.0	.000	.000	.000	
0.1	.083	.085	.087	
0.2	.215	.227	.233	
0.3	.377	.403	.421	
0.4	.560	.603	.638	
0.5	.758	.818	.868	
0.6	.964	1.034	1.093	
0.7	1.170	1.244	1.291	
0.8	1.376	1.436	1.447	
0.9	1.574	1.606	1.542	
1.0	1.764	1.744	1.574	
1.1	1.941	1.853	1.547	
1.2	2.103	1.931	1.472	
1.3	2.250	1.979	1.363	
1.4	2.386	2.003	1.235	
1.5	2.505	2.005	1.102	
1.6	2.612	1.991	.972	
1.7	2.704	1.962	.854	
1.8	2.785	1.927	.751	
1.9	2.850	1.884	.664	
2.0	2.920	1.839	.597	
2.1	2.974	1.793	.547	
2.2	3.021	1.747	.513	
2.3	3.062	1.702	.496	
2.4	3.098	1.659	.492	
2.5	3.129	1.620	.501	
2.6	3.157	1.583	.520	
2.7	3.181	1.550	.547	
2.8	3.204	1.520	.582	
2.9	3.223	1.496	.623	
3.0	3.242	1.476	.669	
3.1				
3.2				
3.3				
3.4				
3.5				
3.6				
3.7				

Table 4. G(u) for E = 0 (parabolic case).

(u + 1)	$v_m = \infty$	$v_m = 2$	$v_m = 1$	$v_m = 1/2$	$v_m = 0$
0.0	.000	.000	.000	.000	
0.1	.080	.082	.084	.085	
0.2	.201	.213	.221	.229	
0.3	.345	.374	.394	.415	
0.4	.506	.562	.596	.637	
0.5	.681	.763	.818	.883	
0.6	.869	.976	1.049	1.136	
0.7	1.067	1.192	1.277	1.372	
0.8	1.275	1.406	1.487	1.558	
0.9	1.492	1.612	1.661	1.663	
1.0	1.717	1.795	1.790	1.664	
1.1	1.950	1.959	1.861	1.557	
1.2	2.190	2.082	1.874	1.359	
1.3	2.436	2.180	1.828	1.102	
1.4	2.689	2.244	1.734	.823	
1.5	2.948	2.276	1.601	.550	
1.6	3.213	2.274	1.441	.311	
1.7	3.484	2.238	1.266		
1.8	3.760	2.182	1.085		
1.9	4.041	2.103	.907		
2.0	4.327	2.007	.738		
2.1		1.899	.582		
2.2		1.778	.442		
2.3		1.658	.310		
2.4		1.531	.208		
2.5		1.404			
2.6		1.279			
2.7		1.157			
2.8		1.040			
2.9		.928			
3.0		.822			
3.1		.724			
3.2		.632			
3.3		.549			
3.4		.472			
3.5		.404			
3.6		.343			
3.7		.289			

2/3 of its initial section. Up to the distance from the cathode at which the minimum section is reached, the potential is quite closely the same as for a plane parallel beam, rising somewhat higher because of the extra charge density due to construction of the beam. Beyond the minimum section the potential depends rather strongly on the parameter E which (see equation 35) measures the ultimate angle of flare of the beam. Large E means relatively small flare, and hence the charge density tends to remain constant giving a necessary potential variation as for a plane parallel beam. As E is diminished, the beam

Table 5. Limiting form of G(u) for plane parallel beam.

$u+1$	$(9/4)^{2/3}(u+1)^{4/3}$	$u+1$	$(9/4)^{2/3}(u+1)^{4/3}$
0.0	0.0	1.6	3.205
0.1	0.0796	1.8	3.750
0.2	0.202	2.0	4.32
0.3	—	2.2	4.900
0.4	0.504	2.4	5.505
0.5	—	2.6	6.130
0.6	0.867	2.8	6.76
0.7	—	3.0	7.42
0.8	1.232	3.2	8.09
0.9	—	3.4	8.76
1.0	1.717	3.6	9.46
1.2	2.060	3.8	10.20
1.4	2.683		

flares more giving greatly reduced charge density with increasing distance from the cathode, and hence the potential does not need to rise as much. From Figure 6 we see that if the beam enters a field-free region beyond u about equal to 1, the beam will flare by an amount corresponding to E a little less than 1/2. To produce even greater flare of the beam, it is then necessary to decelerate the beam as shown by the falling potential curve for $E = 0$.

Qualitatively similar behavior is shown for $v_m = 1$ as seen on Figure 8.

Figure 10 is convenient as an aid in visualizing the relative amounts of flare associated with different values of E.

6. CALCULATION OF ELECTRODE SHAPES

From the results of Section 4, we see that the shape of the potential variation along the axis of a beam depends only on two shape parameters, v_m which measures the fractional contraction of the beam in going from the cathode to the place of minimum section, and E the parameter which determines the degree of flare of the beam at large distances from the cathode relative to the shape of the trajectory at the cathode. From equation 33 we see that the absolute value of the potential depends on the desired initial current density I_0 and the linear scale of the electrode system as measured by the parameter x_m, which is the distance from the cathode to the place of minimum section.

The potential along the axis does not depend on the parameter y_0 which specifies the actual half-width of the cathode, and hence the total current per unit length of the beam.

We now wish to complete the analysis by considering the following problem: What electrode shapes must be used and what potentials must be applied to them in order to produce a beam of given values of v_m, E, x_m, I_0, and $\dot{y}_0$? The answer is obtained as follows:

a) From the table, one obtains a suitable G(u) or, if these tables do not cover the required values of v_m and E closely enough, a suitable numerical integration of equation 34 must be carried out.

b) Then from equation 33, using the given values of I_0 and x_m, one calculates the absolute values of the potential along the axis of the beam, U(u).

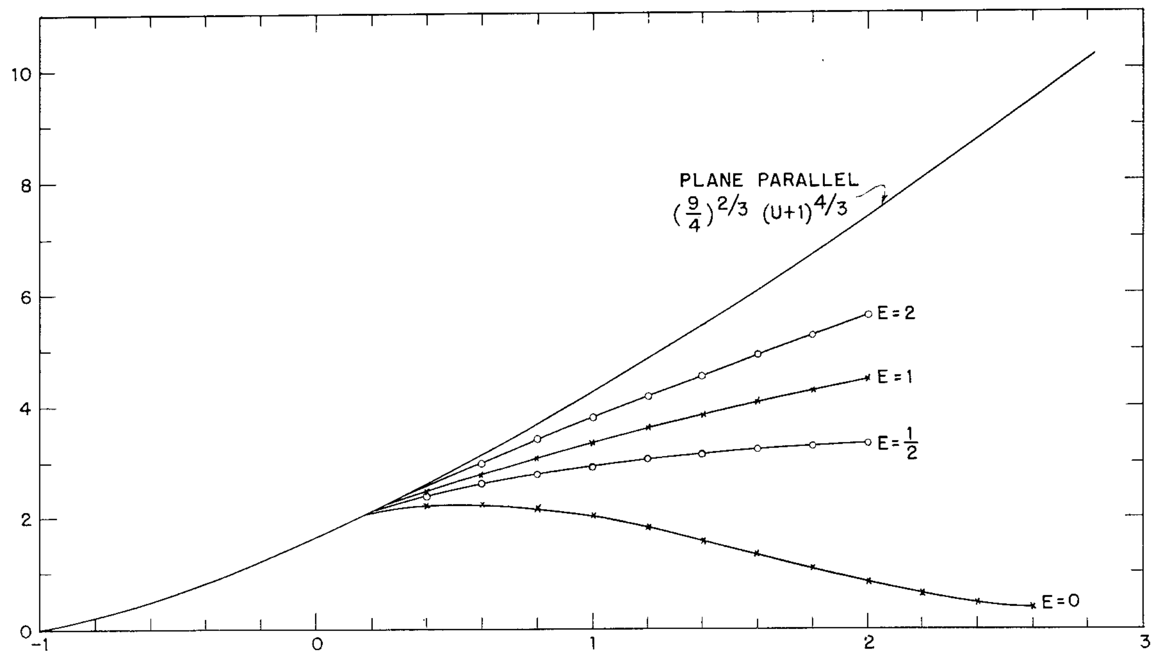

Figure 6. $v_m = 2$. For a beam whose minimum section is 2/3 the initial section. Ordinates: G(u). Flaring increases as E decreases hence less potential is required.

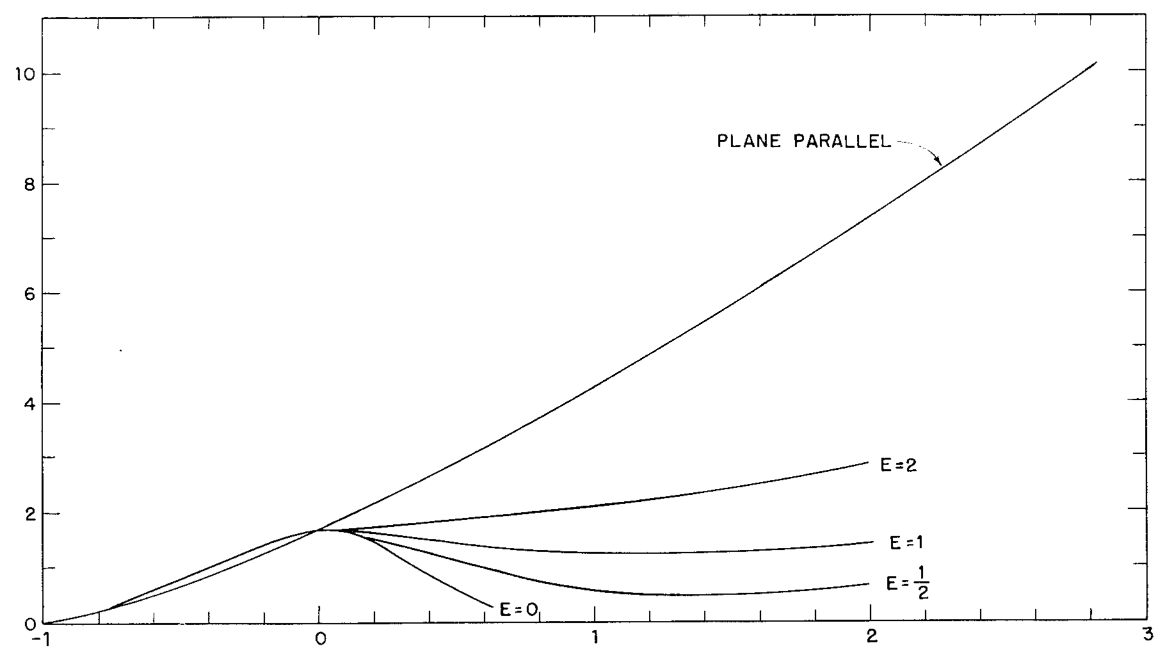

Figure 6a. $v_m = 1/2$. For a beam whose minimum section is 1/3 the initial section. Ordinates: G(u).

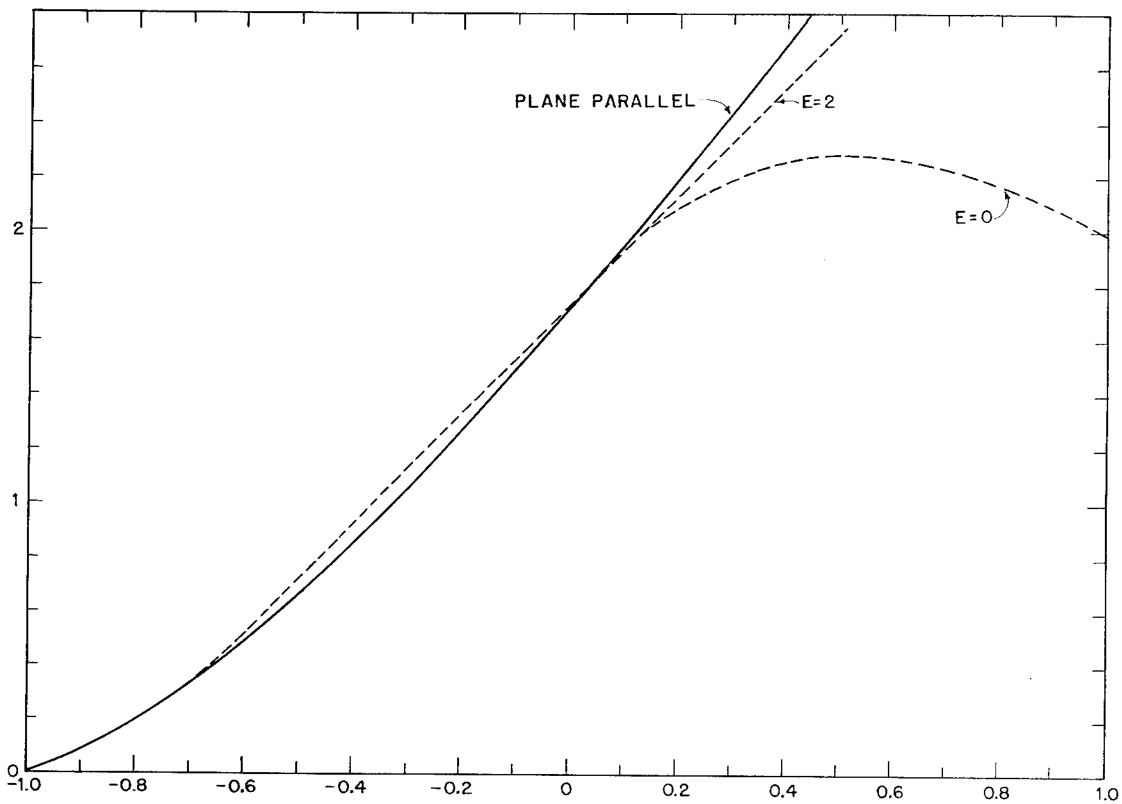

Figure 7. $v_m = 2$. Initial part of same curves as in Figure 6. showing very close agreement with plane parallel case.

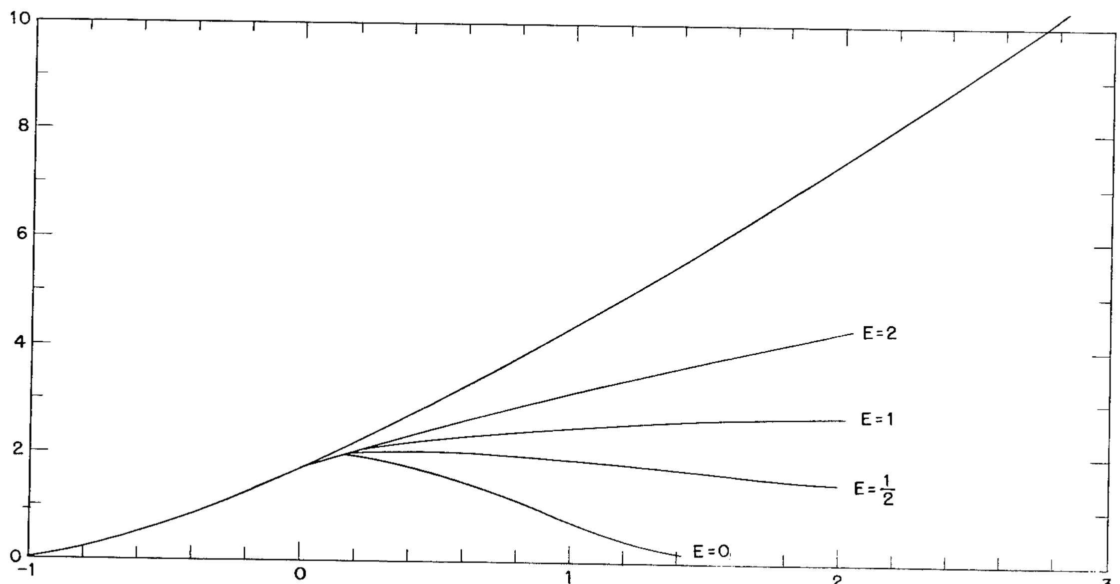

Figure 8. $v_m = 1$. For a beam whose minimum section is 1/2 the initial section. Ordinates: G(u).

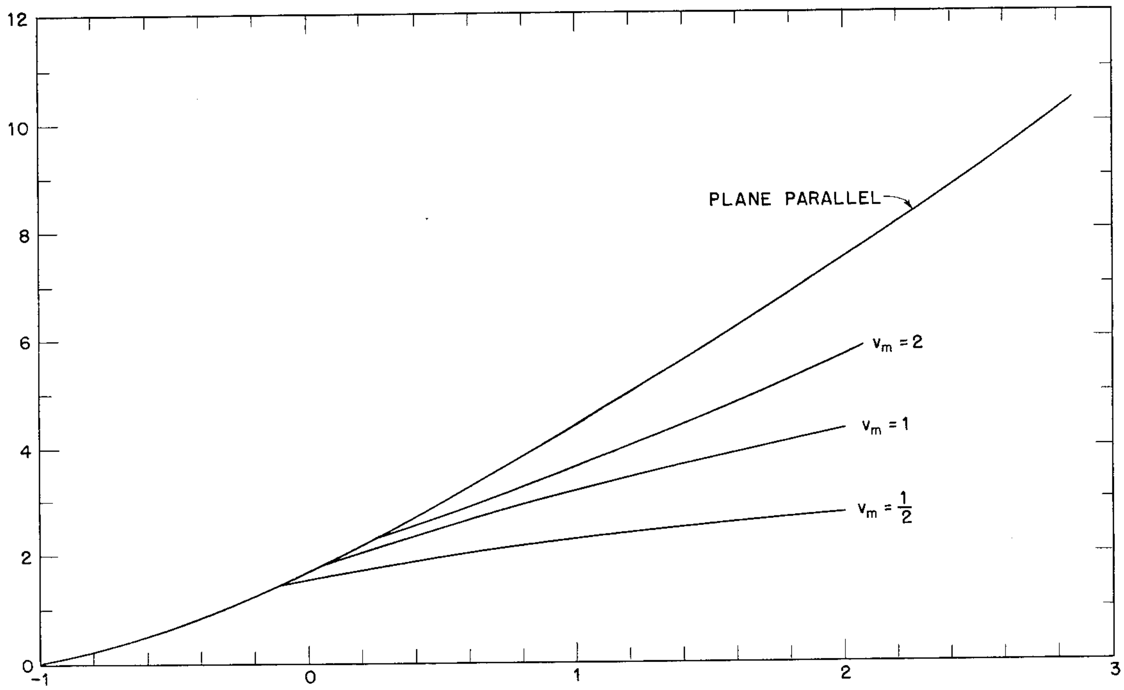

Figure 9. E = 2. Showing effect of varying v_m. Final semi-angle of flare α is $\tan \alpha = \frac{1}{E(v_m + 1)}(y_0/x_m)$.

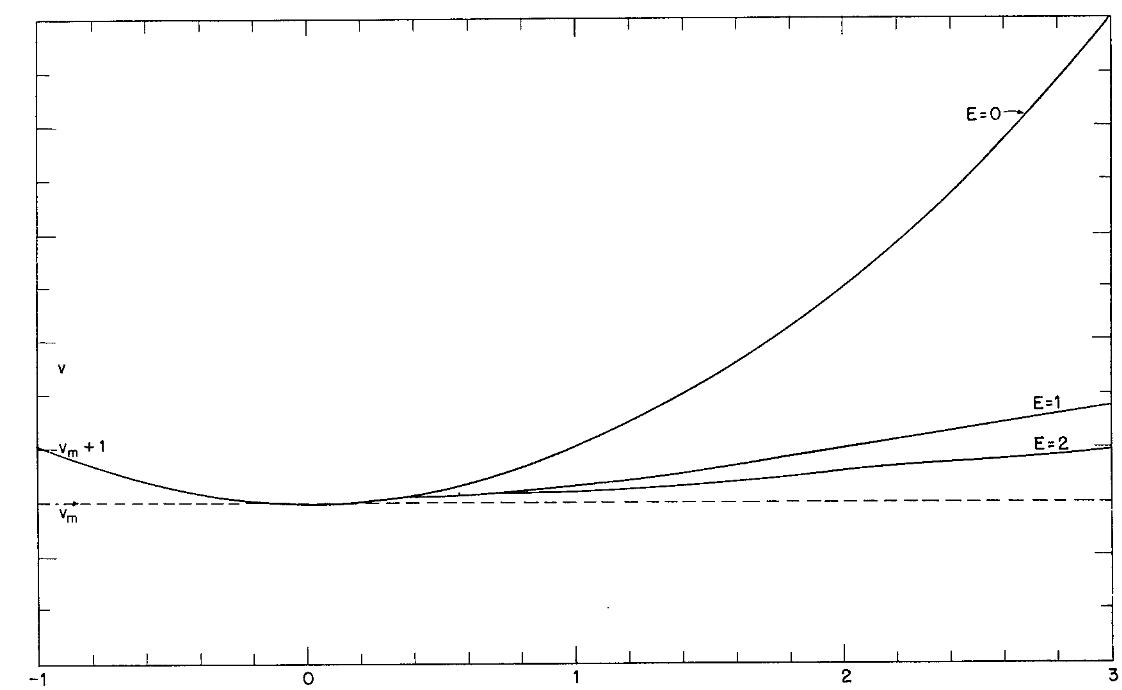

Figure 10. Showing relative amounts of flare for various values of E from equation 31.

c) The potential along the curved edge of the beam and the normal electric gradient along the edge of the beam has then to be calculated by using equation 20 in combination with equation 25, regarded as an equation for W(x). As a result of this calculation, one knows the value of V(x,y) and $\frac{\delta V}{\delta y}$ (x,y) along one edge of the beam.

d) Outside the beam, the potential V(x,y) must satisfy Laplace's equation and the boundary conditions along the edge of the beam $y = Y(x)$ calculated from the preceding step. Any of the known methods of computing or determining experimentally such a solution may be applied for this step, e.g., electrolytic trough or rubber sheet model.

e) The calculation in step (d) will define a series of equipotential surfaces. Various ones of these may be chosen as the actual shape for the electrodes to which, if the calculated potentials are applied, the resulting electric fields will be of the kind required to produce the desired space-charge-limited electron beam.

The actual steps of doing this process will be illustrated, and some detailed calculations presented in a sequel to this report which will be prepared soon.